PRELIMINARY EXAM IN ANALYSIS SPRING 2015

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems. Assume that *measurable* means Lebesgue measurable, unless otherwise stated. We denote the Lebesgue measure by m or dx.

(1) Show that if E is a measurable subset of $[0,1]$ and $m(E)$ is strictly positive then there exist $x, y \in E$ such that $x - y$ is irrational.

(2) The *Bounded Convergence Theorem* is as follows. If f_n is a sequence of measurable functions on a measurable set $E \subset \mathbb{R}^d$ with $m(E) < \infty$ such that $f_n \to f$ pointwise almost everywhere on E and $|f_n| \leq M$ on E then

$$\int_E f_n dx \to \int_E f dx.$$

(a) Using Egorov's Theorem, prove the Bounded Convergence Theorem.
(b) Show that the statement of the Bounded Convergence Theorem is false in general if the hypothesis $m(E) < \infty$ is removed.

(3) Suppose f is real-valued and integrable with respect to the Lebesgue measure m on $\mathbb{R}$. Suppose that there are real numbers $A < B$ such that

$$A\, m(U) \leq \int_U f dx \leq B\, m(U), \quad \text{for all open sets } U \subseteq \mathbb{R}.$$

Show that $A \leq f(x) \leq B$ for almost every x.

(4) Suppose that $f_n \in L^1([0,1], dx)$ converges pointwise almost everywhere to $f \in L^1([0,1], dx)$. Show that f_n converges to f in $L^1([0,1], dx)$ if and only if $\|f_n\|_{L^1} \to \|f\|_{L^1}$.

(5) Let X be a nonempty set.
(a) Define what it means for a map μ_*, from the collection of all subsets of X to $[0, \infty]$, to be an *exterior measure*. (This is also known as an *outer measure*).
(b) Define what it means for a collection $\mathcal{M}$ of subsets of X to be a *σ-algebra*.

We say that a subset $E \subseteq X$ is *Carathéodory measurable* if, for every $A \subseteq X$,

$$\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^c).$$

It is a fact that the set $\mathcal{M}$ of Carathéodory measurable subsets of X is a σ-algebra (you do not need to prove this).

(c) Show that the exterior measure μ_* is a measure when restricted to the collection $\mathcal{M}$ of Carathéodory measurable sets.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) Let $f \in L^1([0,1], dx)$ but $f \notin L^2([0,1], dx)$.
  (a) Show that the set

$$S = \{\phi \in C[0,1] : \int_0^1 \phi(x) f(x) dx = 0\}$$

  is a dense subspace of $L^2([0,1])$.

  (b) Use this density result to show that there exists an orthonormal basis $\{\phi_n\}$ of $L^2([0,1], dx)$ so that $\int_0^1 f\phi_n = 0$ for all n (use density to show that there exists a complete orthonormal basis ϕ_n of $L^2([0,1])$ such that $\phi_n \in S$.)

(2) For $f \in L^2([0,1], dx)$, define the two operators,

$$Sf(x) = \int_0^x f(y)\, dy, \quad Tf(x) = \frac{1}{x} \int_0^x f(y)\, dy.$$

Show that

(a) S is a bounded operator on $L^2([0,1])$.

(b) $S : L^2([0,1]) \to L^2([0,1])$ is a compact operator.

(c) $T : L^2([0,1]) \to L^2([0,1])$ is not a compact operator.

(d) Show that T is a bounded operator on $L^2([0,1])$. (Hint: Assume $f \in C([0,1])$ and write out the integral for $||Tf||^2$. Integrate by parts and apply the Cauchy-Schwarz inequality).

(3) Define the Schwartz space $\mathcal{S}(\mathbb{R})$. Use the Fourier transform (or another method) to show that there exists $f \in \mathcal{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} |f(x)|^2 dx = 1$ but $\int_{\mathbb{R}} x^k f(x) dx = 0$ for every k.

(4) Do the following two problems on $L^p([0,1], dx)$. If true, prove it. If false, exhibit a counter-example.

  (a) True or False: If $f \in L^p([0,1], dx)$ for all p, then $f \in L^\infty([0,1], dx)$.
  (b) True or False: Let $f \in L^\infty([0,1], dx)$. Then $\lim_{p\to\infty} ||f||_{L^p([0,1])} = ||f||_{L^\infty([0,1])}$.

(5) Let X and Y be Banach spaces and let $T : X \to Y$ be an injective bounded linear operator. Let Dom(T) be the domain of T and let Ran(T) be the range of T. Call an operator closed if its graph is closed in $X \times Y$. Show that:

  (a) If Ran(T) is closed, then T^{-1} : Ran(T) $\to$ X is a closed operator and $D(T^{-1}) =$ Ran(T).

  (b) T^{-1} : Ran(T) $\to$ X is bounded if and only if Ran(T) is closed in Y.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Let $\gamma(t) = 2 + e^{2\pi it}$ for $t \in [0,1]$. For each integer n, evaluate:

$$\int_\gamma \left(\frac{z}{z-2}\right)^n \, dz$$

(2) Let f be analytic on an open set containing $\{|z| \leq 1\}$, and suppose that $|f(z)| < 1$ for all $|z| = 1$. Show there is a unique point z_0 with $|z_0| < 1$ and $f(z_0) = z_0$.

(3) Suppose U is a simply connected region in $\mathbb{C}$ and f is analytic.

(a) Give an example of U and f such that $f(U)$ is not simply-connected.

(b) If $f(z) \neq 0$ for all $z \in U$, show that the winding number of $f \circ \gamma$ around 0 is 0 for all closed curves γ in U.

(4) (a) Give an example of a power series in z with radius of convergence 1 that does *not* converge for all $|z| \leq 1$.

(b) Prove that the series

$$\sum_n \frac{z^n}{n^2}$$

has radius of convergence 1 and defines a continuous function on $|z| \leq 1$.

(5) Suppose $\{f_n\}$ is a uniformly bounded sequence of analytic functions on the unit disk $\mathbb{D}$, converging pointwise to a function f. Prove that the convergence is uniform on compact subsets of $\mathbb{D}$ and the function f is analytic.